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NONLINEAR HAMMERSTEIN EQUATIONS AND FUNCTIONS OF BOUNDED RIESZ–MEDVEDEV VARIATION

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Abstract. In this paper we study the solvability of a nonlinear Hammerstein type integral equation in the space of functions of bounded Riesz–Medvedev variation. To this end, we derive a compactness criterion and apply Schauder's fixed point theorem to a suitable operator whose fixed points coincide with the solutions of the integral equation.

1. Statement of the problem

Consider the nonlinear integral equation of Hammerstein type

$$f(s) = \int_a^b k(s,t)h(t,f(t))\,dt + b(s). \tag{1.1}$$

We are interested in conditions on the given functions k, h, and b under which the above equation admits a solution f in some space of functions of generalized bounded variation. To this end, we reformulate (1.1) as usual as a fixed point problem $f = Af$ for the operator

$$Af := KHf + b, \tag{1.2}$$

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where

$$Hf(t) = h(t, f(t)) \tag{1.3}$$

is the (nonlinear) composition operator generated by the function $h\colon [a,b]\times\mathbb{R}\to\mathbb{R}$, and

$$Kg(s) = \int_a^b k(s,t)g(t)\,dt \tag{1.4}$$

is the (linear) integral operator generated by the kernel $k\colon [a,b] \times [a,b] \to \mathbb{R}$. Of course, this idea is very old, and there is a wealth of existence theorems in the literature, obtained in this way by considering the operators (1.3) and (1.4) in the space $C([a,b])$ of continuous functions, or in spaces of measurable functions like the Lebesgue space $L_p([a,b])$ and the Orlicz space $L_\phi([a,b])$. On the other hand, to the best of our knowledge there are only very few papers on the solvability of (1.1) in spaces of functions of (classical or generalized) bounded variation, although these spaces frequently occur in applications. As a sample result, we mention [18], where the author considers BV-solutions of a nonlinear convolution-type Volterra integral equation on the real line. More recently, Bugajewska, Bugajewski et al. [6]–[12] study equation (1.1) more systematically from the viewpoint of variation and prove existence of solutions in the spaces $\mathrm{BV}([a,b])$ (bounded Jordan variation, see [15]), $\mathrm{WBV}_p([a,b])$ (bounded Wiener variation, see [31]), or even $\Lambda\mathrm{BV}([a,b])$ (bounded Waterman variation, [27]–[30]).

The purpose of this paper is to prove existence of solutions of (1.1) in the spaces $\mathrm{BV}_p([a,b])$ of functions of bounded Riesz p-variation or, more generally, $\mathrm{BV}_\phi([a,b])$ of functions of bounded Medvedev ϕ-variation. First we recall the definition and some properties of these spaces and derive a natural compactness criterion. Afterwards we impose some conditions on the given data h and k under which the operator (1.2) is continuous and compact, and leaves a closed ball in the space $\mathrm{BV}_p([a,b])$ or $\mathrm{BV}_\phi([a,b])$ invariant. By means of Schauder's fixed point theorem we obtain then existence of solutions of (1.1). Finally, we briefly sketch how our method could be extended to a larger class of equations if we define a measure of noncompactness in $\mathrm{BV}_p([a,b])$ and replace Schauder's theorem by Darbo's fixed point theorem [13].

2. The Riesz–Medvedev variation

We begin this section recalling the definition of the Riesz variation [24], [25] which contains the Jordan variation as special case. Throughout this paper, we take $[a,b] = [0,1]$ for simplicity of notation.