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# THE LEAST NUMBER OF $n$-PERIODIC POINTS ON TORI CAN BE REALIZED BY A SMOOTH MAP

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Abstract. We give an algebraic proof of the Theorem of Cheng Ye You that the least number of $n$-periodic points, in the continuous homotopy class of a self-map of a torus, can be realized by a smooth map.

## 1. Introduction

The classical Nielsen number $N(f)$, of a continuous self-map of a compact polyhedron $f\colon X \to X$, is a homotopy invariant and a lower bound of fixed points in the homotopy class: i.e. for each $g$ homotopic to $f$ the inequality $\#\mathrm{Fix}(g) \geq N(f)$ holds. This number was generalized by Boju Jiang to a homotopy invariant lower bound of the number of $n$-periodic points: $\#\mathrm{Fix}(g^n) \geq NF_n(f)$ for each $g$ homotopic to the given $f$ [17]. Later it was generalized to $NJD_n(f)$, a lower bound of the number of $n$-periodic points in the smooth homotopy class of a self-map $f\colon M \to M$ of a compact manifold [7]. The obvious inequality $NF_n(f) \leq NJD_n(f)$ often turns out to be sharp [6]. However Cheng Ye You had proved much earlier that in the case of self-maps of a torus the both numbers coincide [20]. This result easily generalizes to all self-maps of nil- and solvmanifolds [14]. The proofs, in [20] and in [14], are geometric and they do not use the above invariants. On the other hand the generalized Nielsen theory was successfully used to show when a similar equality takes place in the case of self-maps of

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compact Lie groups with free fundamental group [15]. Here we will reprove the Theorem of Cheng Ye You for self-maps of tori: we will show that an algebraic condition for the equality is satisfied for self-maps of the torus. In a forthcoming paper we will use this condition to prove when the equality holds for any Lie groups. Since in Nielsen theory simple algebraic methods are effective only in dimensions $\geq 3$ (Wecken Theorem), in the last Section we give elementary proof of this Theorem in low dimensions. Of course the last proof must be again geometric.

In the first Sections we recall basic information on Nielsen periodic point theory. In Sections 4 and 5 we recall conditions for smooth realizability of the number of periodic points: Lemma 4.2 and Theorem 5.2. In Section 6 we show that these conditions are satisfied for all self-maps of a torus of dimension $\geq 3$. In the last Section we consider the low dimensional cases.

Here by smoothness we mean $C^1$, although use of any other class $C^\alpha$, where $\alpha = 1, \ldots, \infty$, leads to the same results, since the sequence of indices $(\mathrm{ind}(f^k; x_0))$ realizable by a $C^1$ map is also realizable by a $C^\infty$ map [8].

## 2. Nielsen fixed and periodic points theory

For the details we send the reader to [17] and [16]. We consider a self-map of a compact connected polyhedron $f \colon X \to X$ and its fixed point set $\mathrm{Fix}(f)$. We define the *Nielsen relation* on this set by: $x \sim y$ if and only if there is a path $\omega$ joining $x$ with $y$ so that $f\omega$ and $\omega$ are fixed end point homotopic.

This relation splits $\mathrm{Fix}(f)$ into *Nielsen classes*. Their set will be denoted by $\mathcal{N}(f)$. We say that a Nielsen class $A$ is *essential* if its fixed point index is nonzero: $\mathrm{ind}(f; A) \neq 0$. The number of essential Nielsen classes is called *Nielsen number* and is denoted $N(f)$. This is a homotopy invariant and moreover it is the lower bound of the number of fixed points in the (continuous) homotopy class: $N(f) \leq \min_{h \sim f} \#\mathrm{Fix}(h)$ [3], [17], [16].

On the other hand we define, the set of *Reidemeister classes* of the map $f$ as the quotient set of the action of the fundamental group $\pi_1 M$ on itself given by $\omega * \alpha = \omega \cdot \alpha \cdot (f_\# \omega)^{-1}$. Here we take as the base point a fixed point $x_0 \in \mathrm{Fix}(f)$. We denote the quotient space by $\mathcal{R}(f)$. There is a natural injection from the set of the Nielsen classes to the set of Reidemeister classes $\mathcal{N}(f) \subset \mathcal{R}(f)$ defined as follows. We choose a point $x$ in the given Nielsen class $A$ and a path $\omega$ from the base point $x_0$ to $x$. Then the loop $\omega * (f\omega)^{-1}$ represents the corresponding Reidemeister class.

We will denote the Reidemeister classes of iterations as $[\alpha]^n \in \mathcal{R}(f^n)$ where $\alpha \in \pi_1 M$.

Now we consider iterations of the map $f$. For fixed natural numbers $l|k$ there is a natural inclusion $\mathrm{Fix}(f^l) \subset \mathrm{Fix}(f^k)$ which induces the map $\mathcal{N}(f^l) \to \mathcal{N}(f^k)$