Configuration spaces and multiple positive solutions to a singularly perturbed elliptic system

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We consider the elliptic system of ℓ equations

$$\begin{cases} -\varepsilon^2 \Delta u_i + u_i = \Big(\sum_{j=1}^{\ell} \beta_{ij} u_j^2 \Big) u_i, \\ u_i \in H_0^1(\Omega), \quad u_i \neq 0, \quad i = 1, \dots, \ell, \end{cases} \tag{$*$}$$

where $\varepsilon > 0$ is a small parameter, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N = 2$ or 3, $\ell \geq 2$, $\beta_{ii} > 0$ and $\beta_{ij} = \beta_{ji} < 0$ if $i \neq j$. The equation in $(*)$ corresponding to $\ell = 1$ is $-\varepsilon^2 \Delta u + u = \beta_{11} u^3$ and for certain Ω, e.g. $\Omega =$ ball, there exists exactly one positive solution. If $\ell = 2$, then the equations in $(*)$ are

$$-\varepsilon^2 \Delta u_i + u_i = (\beta_{ii} u_i^2 + \beta_{ij} u_j^2) u_i, \quad i = 1,\ j = 2 \text{ or } i = 2,\ j = 1,$$

and it is known that for small ε there exist at least 2 positive solutions which concentrate as $\varepsilon \to 0$. A natural question is whether the minimal number of positive solutions increases with ℓ. We show that this is the case and if ε is small, then the number of such solutions is at least ℓ. The proof is by estimating the Lusternik-Schnirelman category of a certain level set of the functional associated with $(*)$ in terms of the category of suitable configuration spaces. A crucial fact is that the category of the configuration space of ℓ points in $\mathbb{R}^N$ equals ℓ (a result due to F. Roth).

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