A complete invariant for shift equivalence for Boolean matrices and finite relations

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Let X be a finite set. We can describe a relation R on X by a Boolean matrix, and conversely, a Boolean matrix yields a relation on X. In both settings, shift equivalence is a natural and important dynamical equivalence relation. It is strictly weaker than conjugacy, and corresponds roughly to "eventual conjugacy" ([1]). For matrices, one use is in the classification of shifts of finite type. For relations, it arises in defining a Conley index for computational approximations of dynamical systems. Classifications of shift equivalence are given in [2, 3, 4]. In this talk, I will present a complete invariant in terms of the period, the induced partial order on recurrent components, and the cohomology class of the relation on those components.

The result is as follows. There exists a least integer $p > 1$ such that there exists an integer $N > 0$ such that $R^{n+p} = R^n$ for all $n \geq N$; we call p the period of R. Additionally, R induces a partial order $R_{\leq}$ on the strongly connected components, determined by which components map to which. Finally, a choice of representatives of the strongly connected components induces a cocycle $\xi \colon R_{\leq} \to \mathcal{L}_p$, where $\mathcal{L}_p$ is the collection of non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$; we denote the cohomology class of ξ by $[\xi]$. Now, for a finite relation R, the triple $(R_{\leq}, p, [\xi])$ is a complete invariant of shift equivalence.

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References

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