The Lyapunov exponent and rigorous computation of expansion in one-dimensional dynamics

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The Lyapunov exponent serves as a quantitative gauge of sensitivity to initial conditions within dynamical systems. A positive Lyapunov exponent is often associated with chaotic dynamics. In this study, we focus on one-dimensional dynamics and propose a methodology that is based on the analysis of weighted directed graphs [4] and uses interval arithmetic. Our approach aims to rigorously compute a lower bound for the expansivity rate that is supposed to correspond to the Lyapunov exponent.

We introduce the developed algorithm extending previous approaches [[1], [2], [3]] and perform numerical experiments checking the impact of various parameters on the results of the method, including computations on intervals of parameters, providing insights into its strengths and weaknesses. Additionally, we conduct a comparative analysis against a non-rigorous numerical approximation of the Lyapunov exponent.

Specifically, we evaluate our approach against a well-known family of quadratic maps, $f_a(x) = a - x^2$. This analysis not only evaluates the precision and validity of our computational framework but also sheds light on the limitations of a rigorous interval-based approach in contrast to non-rigorous numerical simulations when estimating the Lyapunov exponent.

References

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