Representations of Hamilton-Jacobi equations in optimal control theory for superlinear Hamiltonians

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The Hamilton-Jacobi equation

$$\begin{aligned} -V_t + H(t,x,-V_x) = 0 \quad &\text{in} \quad (0,T)\times\mathbb{R}^n, \\ V(T,x) = g(x) \quad &\text{in} \quad \mathbb{R}^n, \end{aligned} \tag{1}$$

with a convex Hamiltonian H in the gradient variable can be studied with connection to optimal control problems. It is possible, provided that there exists a sufficiently regular triple (A, f, l) satisfying the following equality

$$H(t,x,p) = \sup_{a\in A} \{ \langle p, f(t,x,a) \rangle - l(t,x,a) \}. \tag{2}$$

Then the value function of the optimal control problem defined by

$$V(t_0,x_0) = \inf_{(x,a)(\cdot)\,\in\, S_f(t_0,x_0)} \Big\{ g(x(T)) + \int_{t_0}^{T} l(t,x(t),a(t))\,dt \Big\}$$

represents the equation (1), where $S_f(t_0, x_0)$ denotes the set of all trajectory-control pairs of the control system $\dot{x}(t) = f(t, x(t), a(t))$, with $a(t) \in A$, for a.e. $t \in [t_0, T]$ and $x(t_0) = x_0$.

The triple (A, f, l), which satisfies the equality (2), is called a representation of H. In the literature (see [1], [2], [3], [4]), one can find constructions of representations for sublinear Hamiltonians (e.g., $H(t, x, p) = \alpha(t)|x||p|$). We demonstrate the construction of representations for superlinear Hamiltonians (e.g., $H(t, x, p) = \alpha(t)|x|^2|p|^2$). It is known that these representations are constructed using theorems on Lipschitz parametrizations of Lipschitz set-valued maps. In our case, we parametrize the set-valued map $E(t, x) = \mathrm{epi}H^*(t, x, \cdot)$ which is derived from the epigraph of the Legendre-Fenchel conjugate of $H(t, x, \cdot)$. While for sublinear Hamiltonians, the set-valued maps are Lipschitz continuous in the Hausdorff sense, for superlinear Hamiltonians, they exhibit Lipschitz continuity in the Aubin sense. The lack of parametrization theorems for the latter case is addressed by introducing an appropriate theorem for parametrizing set-valued maps with Lipschitz continuity in the Aubin sense.

References

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